Central extensions of nilpotent Lie superalgebras

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MAThEOR Days

11th July 2024

Based on a joint work with S. Bouarroudj

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Lie superalgebras

Super vector spaces.

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Definition

A Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space equipped with a bilinear map $[\cdot,\cdot] : L \times L \to L$ satisfying for all $x, y, z \in L$:

1. $|[x,y]| = |x| + |y|$;
2. $[x,y] = -(-1)^{|x||y|}[y,x]$;
3. $(-1)^{|x||z|}[x,[y,z]] + (-1)^{|x||y|}[y,[z,x]] + (-1)^{|y||z|}[z,[x,y]] = 0.$

Let $f : V \to W$ be a map between $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces. Then:

- the map f is called **even** if $f(V_{\bar{i}}) \subset W_{\bar{i}}$;
- the map f is called **odd** if $f(V_{\bar{i}}) \subset W_{\overline{i+1}}$;

Modules of Lie superalgebras

Let L_1, L_2 be Lie superalgebras. A **morphism of Lie superalgebras** is an even linear map $f : L_1 \to L_2$ such that

$$f([x, y]_1) = [f(x), f(y)]_2, \ \forall x, y \in L_1.$$

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Let L a Lie superalgebra. A vector space M is called L-**module** if there is a Lie superalgebras map

$$\phi : L \to \mathrm{End}(M).$$

In particular, we have

$$\phi([x, y]) = \phi(x) \circ \phi(y) - (-1)^{|x||y|} \phi(y) \circ \phi(x).$$

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Examples: trivial and adjoint.

Nilpotent Lie superalgebras

Let L be a Lie superalgebra. We define a **descending central sequence** by

$$C^0(L) = L, \quad \text{and} \quad C^{k+1}(L) = \left[C^k(L), L\right].$$

The Lie superalgebra L is called **nilpotent** if there exists $k \geq 0$ such that $C^k(L) = 0$.

That is, L is nilpotent if there exists $k \geq 0$ such that

$$[...[x_1, x_2], x_3], ...,], x_k] = 0, \quad \forall x_i \in L.$$

Central extensions of Lie superalgebras

Let $(L, [\cdot,\cdot]_L)$ be a Lie superalgebra and let M be a trivial L-module. We consider $E := L \oplus M$ and we define

$$[x+m, y+n]_E := [x,y]_L + \Delta(x,y), \ \forall x,y \in L, \ \forall m,n \in M,$$

with $\Delta : L \times L \to M$ a bilinear super-skewsymmetric map.

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Proposition

The pair $(E, [\cdot,\cdot]_E)$ is a Lie superalgebra if and only if

$$(-1)^{|x||z|}\Delta(x,[y,z]) + (-1)^{|y||x|}\Delta(y,[z,x]) + (-1)^{|z||y|}\Delta(z,[x,y]) = 0. \qquad (1)$$

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The elements Δ that satisfies Eq. (1) are called (Chevalley-Eilenberg) **2-cocycles** with trivial coefficients.

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The elements Δ that satisfies Eq. (1) are called (Chevalley-Eilenberg) **2-cocycles** with trivial coefficients.

In the case where Eq. (1) is satisfied, $(E, [\cdot,\cdot]_E)$ is called **central extension** of L by M.

Central extensions of restricted Lie superalgebras, revisited

Let $(L, [\cdot, \cdot])$ be a Lie superalgebra, and M be an abelian Lie superalgebra (*i.e*, $[m, n] = 0 \; \forall m, n \in M$).

An **extension** of L by M is a short exact sequence of Lie superalgebras

$$0 \longrightarrow M \xrightarrow{\iota} E \xrightarrow{\pi} L \longrightarrow 0.$$

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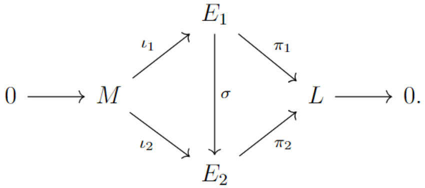
An **extension** of L by M is a short exact sequence of Lie superalgebras

$$0 \longrightarrow M \xrightarrow{\iota} E \xrightarrow{\pi} L \longrightarrow 0.$$

In the case where $\iota(M) \subset \mathfrak{z}(E) := \{a \in E,\ [a, b] = 0 \ \forall b \in E\}$, M is a trivial L-module. These extensions are called **central extensions**.

Two central extensions of L by M are called **equivalent** if there is a Lie superalgebras morphism $\sigma : E_1 \to E_2$ such that the following diagram commutes:



Classification of low dimensional Lie superalgebras

<u>Goal:</u> classify (up to isomorphism) nilpotent Lie superalgebras of dimension 4, over an (alg. closed) field of characteristic **different from 2**.

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Proposition

Let L be a nilpotent Lie superalgebra of dimension n. Then, L is isomorphic to a central extension by a 2-cocycle of a nilpotent Lie superalgebra of dimension $n-1$ by a trivial module M of dimension 1 (i.e., $M = \mathbb{K}$).

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Let L be a nilpotent Lie superalgebra of dimension n. Then, L is isomorphic to a central extension by a 2-cocycle of a nilpotent Lie superalgebra of dimension $n-1$ by a trivial module M of dimension 1 (i.e., $M = \mathbb{K}$).

1. We start from the classification of 3-dimensional nilpotent Lie superalgebras.
2. For each nilpotent 3-dimensional Lie superalgebra, we compute the equivalence classes of 2-cocycles under the action by automorphisms given by
$$(A \cdot \Delta)(x, y) = \Delta\big(A(x), A(y)\big), \ \forall x, y \in L \qquad (2)$$
3. We build the corresponding central extensions.
4. Some of the superalgebras obtained are isomorphic. We detect and remove redundancies.

Dimension 3 (by brute force)

- sdim$(L) = (0|3)$: $L = \langle 0|e_1, e_2, e_3\rangle$: $[\cdot,\cdot] \equiv 0$.
- sdim$(L) = (1|2)$: $L = \langle e_1|e_2, e_3\rangle$.
  1. $\mathbf{L}^1_{1|2} = \langle e_1|e_2, e_3\rangle$ (abelian):
  2. $\mathbf{L}^2_{1|2} = \langle e_1|e_2, e_3; [e_2, e_3] = e_1\rangle$:
  3. $\mathbf{L}^3_{1|2} = \langle e_1|e_2, e_3; [e_1, e_2] = e_3\rangle$:
  4. $\mathbf{L}^4_{1|2} = \langle e_1|e_2, e_3; [e_3, e_3] = e_1\rangle$:
- sdim$(L) = (2|1)$: $L = \langle e_1, e_2|e_3\rangle$.
  1. $\mathbf{L}^1_{2|1} = \langle e_1, e_2|e_3\rangle$ (abelian):
  2. $\mathbf{L}^2_{2|1} = \langle e_1, e_2|e_3; [e_3, e_3] = e_2\rangle$:
- sdim$(L) = (3|0)$: $L = \langle e_1, e_2, e_3\rangle$.
  1. $\mathbf{L}^1_{3|0} = \langle e_1, e_2, e_3\rangle$ (abelian):
  2. $\mathbf{L}^2_{3|0} = \langle e_1, e_2, e_3; [e_1, e_2] = e_3\rangle$

Dimension 4: scalar restricted 2-cocycles

Notation: Let $L = L_{\bar{0}} \oplus L_{\bar{1}} = \langle e_1, \cdots, e_n | e_{n+1}, \cdots, e_{n+m} \rangle$ be a Lie superalgebra of $\mathrm{sdim}(L) = (n|m)$. A basis for the scalar 2-cocycles is then given by

$$\Delta_{i,j} : L \times L \longrightarrow \mathbb{K}, \qquad 1 \leq i \leq n+m, \ i \leq j \leq n+m,$$

where $\Delta_{i,j}(e_k, e_l) = \delta_{i,k}\delta_{j,l}$ and $\Delta_{i,j} = -(-1)^{|e_i||e_j|}\Delta_{j,i}$.

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Theorem

Suppose that L is a nilpotent Lie superalgebra of total dimension 3 with $\dim(L_{\bar{1}}) \geq 1$ over an algebraically closed field of characteristic $p \geq 3$. The equivalence classes of non trivial homogeneous 2-cocycles on L are given by

$L = \mathbf{L}^1_{0|3}$: $\Delta_{1,1}, \ \ \Delta_{1,2}, \ \Delta_{1,1} + \Delta_{2,3}$;

$L = \mathbf{L}^1_{1|2}$: $\Delta_{1,2}, \ \Delta_{2,3}, \ \Delta_{2,2} + \Delta_{2,3} + \Delta_{3,3}$;

$L = \mathbf{L}^2_{1|2}$: $\Delta_{2,2}, \ \Delta_{2,2} + \Delta_{3,3}$;

$L = \mathbf{L}^3_{1|2}$: $\Delta_{1,3}, \ \Delta_{2,2}$;

$L = \mathbf{L}^4_{1|2}$: $\Delta_{2,2}, \ \Delta_{2,3}, \ \Delta_{2,2} + \Delta_{2,3}$.

$L = \mathbf{L}^1_{2|1}$: $\Delta_{1,3}, \ \Delta_{1,2}, \ \Delta_{3,3}, \ \Delta_{1,2} + \Delta_{3,3}$;

$L = \mathbf{L}^2_{2|1}$: $\Delta_{1,3}$.

Dimension 4: the classification. Building the extensions.

With the list of 2-cocycles, we can extend the Lie brackets using

$$[x, y]_{\text{new}} = [x, y]_{\text{old}} + \Delta(x, y)X. \tag{3}$$

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Example. Consider $\mathbf{L^3_{1|2}} = \langle e_1 | e_2, e_3; [e_1, e_2] = e_3 \rangle$. The 2-cocycles are $\Delta_{1,3}$ and $\Delta_{2,2}$. We obtain four superalgebras of dimension 4.

Name	sdim	Cocycle	Added element	Bracket
$\mathbf{L^g_{2\|2}}$	$(2\|2)$	0	X even	$[e_1, e_2] = e_3$
$\mathbf{L^d_{1\|3}}$	$(1\|3)$	0	X odd	$[e_1, e_2] = e_3$
$\mathbf{L^e_{1\|3}}$	$(1\|3)$	$\Delta_{1,3}$	X odd	$[e_1, e_2] = e_3,\ [e_1, e_3] = X$
$\mathbf{L^h_{2\|2}}$	$(2\|2)$	$\Delta_{2,2}$	X even	$[e_1, e_2] = e_3,\ [e_2, e_2] = X$

Lie superalgebras obtained by central extensions of $\mathbf{L^3_{1|2}}$.

Dimension 4: the classification. Building the extensions.

Name	sdim	Cocycle	Added element	Bracket
$\mathbf{L^g_{2\|2}}$	$(2\|2)$	0	X even	$[e_1, e_2] = e_3$
$\mathbf{L^d_{1\|3}}$	$(1\|3)$	0	X odd	$[e_1, e_2] = e_3$
$\mathbf{L^e_{1\|3}}$	$(1\|3)$	$\Delta_{1,3}$	X odd	$[e_1, e_2] = e_3, \ [e_1, e_3] = X$
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Lie superalgebras obtained by central extensions of $\mathbf{L^3_{1|2}}$.

Name	sdim	Cocycle	Added element	Bracket
$\mathbf{L^a_{2\|2}}$	$(2\|2)$	0	X even	$[\cdot, \cdot] = 0$
$\mathbf{L^a_{1\|3}}$	$(1\|3)$	0	X odd	$[\cdot, \cdot] = 0$
$\mathbf{L^b_{1\|3}}$	$(1\|3)$	$\Delta_{1,2}$	X odd	$[e_1, e_3] = X$
$\mathbf{L^b_{2\|2}}$	$(2\|2)$	$\Delta_{2,3}$	X even	$[e_2, e_3] = X$
$\mathbf{L^c_{2\|2}}$	$(2\|2)$	$\Delta_{2,2} + \Delta_{2,3} + \Delta_{3,3}$	X even	$[e_2, e_2] = [e_2, e_3] = [e_3, e_3] = X$

Lie superalgebras obtained by extensions of $\mathbf{L^1_{1|2}}$ (abelian).

Dimension 4: the classification. Building the extensions.

Name	sdim	Cocycle	Added element	Bracket
$\mathbf{L}^{\mathbf{g}}_{\mathbf{2\|2}}$	$(2\|2)$	0	X even	$[e_1, e_2] = e_3$
$\mathbf{L}^{\mathbf{d}}_{\mathbf{1\|3}}$	$(1\|3)$	0	X odd	$[e_1, e_2] = e_3$
$\mathbf{L}^{\mathbf{e}}_{\mathbf{1\|3}}$	$(1\|3)$	$\Delta_{1,3}$	X odd	$[e_1, e_2] = e_3, \ [e_1, e_3] = X$
$\mathbf{L}^{\mathbf{h}}_{\mathbf{2\|2}}$	$(2\|2)$	$\Delta_{2,2}$	X even	$[e_1, e_2] = e_3, \ [e_2, e_2] = X$

Lie superalgebras obtained by central extensions of $\mathbf{L}^{\mathbf{3}}_{\mathbf{1|2}}$.

Name	sdim	Cocycle	Added element	Bracket
$\mathbf{L}^{\mathbf{a}}_{\mathbf{2\|2}}$	$(2\|2)$	0	X even	$[\cdot, \cdot] = 0$
$\mathbf{L}^{\mathbf{a}}_{\mathbf{1\|3}}$	$(1\|3)$	0	X odd	$[\cdot, \cdot] = 0$
$\mathbf{L}^{\mathbf{b}}_{\mathbf{1\|3}}$	$(1\|3)$	$\Delta_{1,2}$	X odd	$[e_1, e_3] = X$
$\mathbf{L}^{\mathbf{b}}_{\mathbf{2\|2}}$	$(2\|2)$	$\Delta_{2,3}$	X even	$[e_2, e_3] = X$
$\mathbf{L}^{\mathbf{c}}_{\mathbf{2\|2}}$	$(2\|2)$	$\Delta_{2,2} + \Delta_{2,3} + \Delta_{3,3}$	X even	$[e_2, e_2] = [e_2, e_3] = [e_3, e_3] = X$

Lie superalgebras obtained by extensions of $\mathbf{L}^{\mathbf{1}}_{\mathbf{1|2}}$ (abelian).

We know that $\mathbf{L}^{\mathbf{d}}_{\mathbf{1|3}} \not\cong \mathbf{L}^{\mathbf{e}}_{\mathbf{1|3}}$ and $\mathbf{L}^{\mathbf{a}}_{\mathbf{1|3}} \not\cong \mathbf{L}^{\mathbf{b}}_{\mathbf{1|3}}$...

Dimension 4: the classification. Building the extensions.

Name	sdim	Cocycle	Added element	Bracket
$\mathbf{L}^{\mathbf{g}}_{\mathbf{2\|2}}$	$(2\|2)$	0	X even	$[e_1, e_2] = e_3$
$\mathbf{L}^{\mathbf{d}}_{\mathbf{1\|3}}$	$(1\|3)$	0	X odd	$[e_1, e_2] = e_3$
$\mathbf{L}^{\mathbf{e}}_{\mathbf{1\|3}}$	$(1\|3)$	$\Delta_{1,3}$	X odd	$[e_1, e_2] = e_3,\ [e_1, e_3] = X$
$\mathbf{L}^{\mathbf{h}}_{\mathbf{2\|2}}$	$(2\|2)$	$\Delta_{2,2}$	X even	$[e_1, e_2] = e_3,\ [e_2, e_2] = X$

Lie superalgebras obtained by central extensions of $\mathbf{L}^{\mathbf{3}}_{\mathbf{1|2}}$.

Name	sdim	Cocycle	Added element	Bracket
$\mathbf{L}^{\mathbf{a}}_{\mathbf{2\|2}}$	$(2\|2)$	0	X even	$[\cdot, \cdot] = 0$
$\mathbf{L}^{\mathbf{a}}_{\mathbf{1\|3}}$	$(1\|3)$	0	X odd	$[\cdot, \cdot] = 0$
$\mathbf{L}^{\mathbf{b}}_{\mathbf{1\|3}}$	$(1\|3)$	$\Delta_{1,2}$	X odd	$[e_1, e_3] = X$
$\mathbf{L}^{\mathbf{b}}_{\mathbf{2\|2}}$	$(2\|2)$	$\Delta_{2,3}$	X even	$[e_2, e_3] = X$
$\mathbf{L}^{\mathbf{c}}_{\mathbf{2\|2}}$	$(2\|2)$	$\Delta_{2,2} + \Delta_{2,3} + \Delta_{3,3}$	X even	$[e_2, e_2] = [e_2, e_3] = [e_3, e_3] = X$

Lie superalgebras obtained by extensions of $\mathbf{L}^{\mathbf{1}}_{\mathbf{1|2}}$ (abelian).

We know that $\mathbf{L}^{\mathbf{d}}_{\mathbf{1|3}} \not\cong \mathbf{L}^{\mathbf{e}}_{\mathbf{1|3}}$ and $\mathbf{L}^{\mathbf{a}}_{\mathbf{1|3}} \not\cong \mathbf{L}^{\mathbf{b}}_{\mathbf{1|3}}$...

But $\mathbf{L}^{\mathbf{b}}_{\mathbf{1|3}} \cong \mathbf{L}^{\mathbf{d}}_{\mathbf{1|3}}$.

Dimension 4: the classification. Detecting isomorphisms.

It is possible that two superalgebras obtained as central extensions by non-equivalent cocycles are isomorphic. We need to detect and remove redundancies.

L	$[L, L]$	$\mathrm{sdim}(\mathfrak{z}(L))$	$\mathrm{sdim}\left(H^1_{\mathrm{CE}}(L;\mathbb{K})\right)$	$\mathrm{sdim}\left(H^2_{\mathrm{CE}}(L;\mathbb{K})\right)$	$\mathrm{sdim}\left(H^3_{\mathrm{CE}}(L;\mathbb{K})\right)$
$\mathbf{L}^{\mathbf{a}}_{\mathbf{1}\mid\mathbf{3}}$	0	$1\mid 3$	$1\mid 3$	$6\mid 3$	$7\mid 9$
$\mathbf{L}^{\mathbf{b}}_{\mathbf{1}\mid\mathbf{3}}$	$\langle X\rangle$	$0\mid 2$	$1\mid 2$	$3\mid 2$	$3\mid 4$ ($3\mid 5$ if $p = 3$)
$\mathbf{L}^{\mathbf{c}}_{\mathbf{1}\mid\mathbf{3}}$	$\langle e_1\rangle$	$1\mid 1$	$0\mid 3$	$5\mid 0$	$0\mid 7$
$\mathbf{L}^{\mathbf{d}}_{\mathbf{1}\mid\mathbf{3}}$	$\langle X\rangle$	$0\mid 2$	$1\mid 2$	$3\mid 2$	$3\mid 4$ ($3\mid 5$ if $p = 3$)
$\mathbf{L}^{\mathbf{e}}_{\mathbf{1}\mid\mathbf{3}}$	$\langle e_3, X\rangle$	$0\mid 1$	$1\mid 1$	$2\mid 1$	$2\mid 2$ ($2\mid 4$ if $p = 3$)
$\mathbf{L}^{\mathbf{f}}_{\mathbf{1}\mid\mathbf{3}}$	$\langle e_1\rangle$	$1\mid 2$	$0\mid 3$	$5\mid 0$	$0\mid 7$
$\mathbf{L}^{\mathbf{j}}_{\mathbf{1}\mid\mathbf{3}}$	$\langle X\rangle$	$1\mid 0$	$0\mid 3$	$5\mid 0$	$0\mid 7$

Invariants for Lie superalgebras of $\mathrm{sdim} = (1\mid 3)$.

Dimension 4: the classification. Lie superalgebras.

Theorem

The classification of 4-dimensional nilpotent Lie superalgebras over an algebraically closed field of characteristic different from 2 is given by:

$\underline{sdim(L) = (0|4)}$: $L = \langle 0|x_1, x_2, x_3, x_4\rangle$

$\mathbf{L}^{\mathbf{1}}_{\mathbf{0|4}}$: $[\cdot,\cdot] = 0$.

$\underline{sdim(L) = (1|3)}$: $L = \langle x_1|x_2, x_3, x_4\rangle$

$\mathbf{L}^{\mathbf{1}}_{\mathbf{1|3}}$ $(= \mathbf{L}^{\mathbf{a}}_{\mathbf{1|3}})$: *abelian;*

$\mathbf{L}^{\mathbf{2}}_{\mathbf{1|3}}$ $(= \mathbf{L}^{\mathbf{b}}_{\mathbf{1|3}})$: $[x_1, x_3] = x_4$;

$\mathbf{L}^{\mathbf{3}}_{\mathbf{1|3}}$ $(= \mathbf{L}^{\mathbf{c}}_{\mathbf{1|3}})$: $[x_2, x_3] = x_1$;

$\mathbf{L}^{\mathbf{4}}_{\mathbf{1|3}}$ $(= \mathbf{L}^{\mathbf{e}}_{\mathbf{1|3}})$: $[x_1, x_2] = x_3$, $[x_1, x_3] = x_4$;

$\mathbf{L}^{\mathbf{5}}_{\mathbf{1|3}}$ $(= \mathbf{L}^{\mathbf{f}}_{\mathbf{1|3}})$: $[x_3, x_3] = x_1$;

$\mathbf{L}^{\mathbf{6}}_{\mathbf{1|3}}$ $(= \mathbf{L}^{\mathbf{j}}_{\mathbf{1|3}})$: $[x_2, x_2] = x_1$, $[x_3, x_4] = x_1$.

$\underline{sdim(L) = (2|2)}$: $L = \langle x_1, x_2|x_3, x_4\rangle$

$\mathbf{L}^{\mathbf{1}}_{\mathbf{2|2}}$ $(= \mathbf{L}^{\mathbf{a}}_{\mathbf{2|2}})$: *abelian;*

$\mathbf{L}^{\mathbf{2}}_{\mathbf{2|2}}$ $(= \mathbf{L}^{\mathbf{b}}_{\mathbf{2|2}})$: $[x_3, x_4] = x_2$;

$\mathbf{L}^{\mathbf{3}}_{\mathbf{2|2}}$ $(= \mathbf{L}^{\mathbf{e}}_{\mathbf{2|2}})$: $[x_3, x_3] = x_2$, $[x_3, x_4] = x_1$;

$\mathbf{L}^{\mathbf{4}}_{\mathbf{2|2}}$ $(= \mathbf{L}^{\mathbf{f}}_{\mathbf{2|2}})$: $[x_3, x_3] = [x_4, x_4] = x_2$, $[x_3, x_4] = x_1$;

$\mathbf{L}^{\mathbf{5}}_{\mathbf{2|2}}$ $(= \mathbf{L}^{\mathbf{g}}_{\mathbf{2|2}})$: $[x_1, x_3] = x_4$;

$\mathbf{L}^{\mathbf{6}}_{\mathbf{2|2}}$ $(= \mathbf{L}^{\mathbf{h}}_{\mathbf{2|2}})$: $[x_1, x_3] = x_4$, $[x_3, x_3] = x_2$.

$\mathbf{L}^{\mathbf{7}}_{\mathbf{2|2}}$ $(= \mathbf{L}^{\mathbf{i}}_{\mathbf{2|2}})$: $[x_4, x_4] = x_1$.

$\underline{sdim(L) = (3|1)}$: $L = \langle x_1, x_2, x_3|x_4\rangle$

$\mathbf{L}^{\mathbf{1}}_{\mathbf{3|1}}$ $(= \mathbf{L}^{\mathbf{a}}_{\mathbf{3|1}})$: *abelian;*

$\mathbf{L}^{\mathbf{2}}_{\mathbf{3|1}}$ $(= \mathbf{L}^{\mathbf{b}}_{\mathbf{3|1}})$: $[x_1, x_2] = x_3$;

$\mathbf{L}^{\mathbf{3}}_{\mathbf{3|1}}$ $(= \mathbf{L}^{\mathbf{c}}_{\mathbf{3|1}})$: $[x_2, x_2] = x_3$;

$\mathbf{L}^{\mathbf{4}}_{\mathbf{3|1}}$ $(= \mathbf{L}^{\mathbf{d}}_{\mathbf{3|1}})$: $[x_1, x_2] = [x_3, x_4] = x_3$.

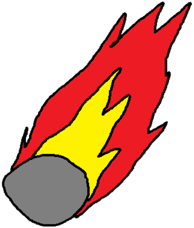
$\underline{sdim(L) = (4|0)}$: $L = \langle x_1, x_2, x_3, x_4|0\rangle$

$\mathbf{L}^{\mathbf{1}}_{\mathbf{4|0}}$: *abelian;*

$\mathbf{L}^{\mathbf{2}}_{\mathbf{4|0}}$: $[x_1, x_2] = x_3$;

$\mathbf{L}^{\mathbf{3}}_{\mathbf{4|0}}$: $[x_1, x_2] = x_3$, $[x_1, x_3] = x_4$.

Thank you for your attention!



Main reference:

S. Bouarroudj, Q. Ehret, *Central extensions of restricted Lie superalgebras and classification of p-nilpotent Lie superalgebras in dimension 4*, January 2024, `arXiv:2401.08313`.